



MATHEMATICS HIGHER LEVEL PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Thursday 8 November 2012 (morning)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the *Mathematics HL and Further Mathematics SL* information booklet is required for this paper.
- The maximum mark for this examination paper is [60 marks].

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [*Maximum mark: 13*]

A differential equation is given by $\frac{dy}{dx} = \frac{y}{x}$, where x > 0 and y > 0.

(a) Solve this differential equation by separating the variables, giving your answer in the form y = f(x).

[3 marks]

(b) Solve the same differential equation by using the standard homogeneous substitution y = vx.

[4 marks]

(c) Solve the same differential equation by the use of an integrating factor.

[5 marks]

(d) If y = 20 when x = 2, find y when x = 5.

[1 mark]

2. [Maximum mark: 12]

Let the differential equation $\frac{dy}{dx} = \sqrt{x+y}$, $(x+y \ge 0)$ satisfying the initial conditions y=1 when x=1. Also let y=c when x=2.

(a) Use Euler's method to find an approximation for the value of c, using a step length of h = 0.1. Give your answer to four decimal places.

[6 marks]

You are told that if Euler's method is used with h = 0.05 then c = 2.7921, if it is used with h = 0.01 then c = 2.8099 and if it is used with h = 0.005 then c = 2.8121.

(b) Plot on graph paper, with h on the horizontal axis and the approximation for c on the vertical axis, the four points (one of which you have calculated and three of which have been given). Use a scale of 1 cm = 0.01 on both axes. Take the horizontal axis from 0 to 0.12 and the vertical axis from 2.76 to 2.82.

[3 marks]

(c) Draw, by eye, the straight line that best fits these four points, using a ruler.

[1 mark]

(d) Use your graph to give the best possible estimate for c, giving your answer to three decimal places.

[2 marks]

3. [Maximum mark: 17]

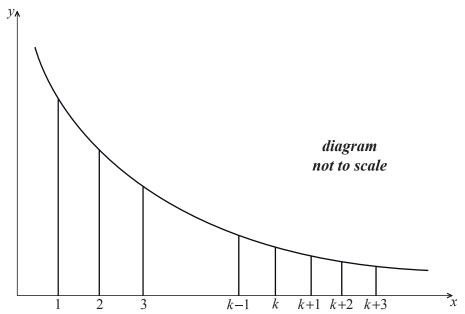
(a) Prove that $\lim_{H \to \infty} \int_a^H \frac{1}{x^2} dx$ exists and find its value in terms of a (where $a \in \mathbb{R}^+$). [3 marks]

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(b) Use the integral test to prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. [3 marks]

Let
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = L.$$

(c) The diagram below shows the graph of $y = \frac{1}{x^2}$.



- (i) Shade suitable regions on a copy of the diagram above and show that $\sum_{k=1}^{k} \frac{1}{n^2} + \int_{k+1}^{\infty} \frac{1}{x^2} dx < L.$
- (ii) Similarly shade suitable regions on another copy of the diagram above and show that $L < \sum_{n=1}^{k} \frac{1}{n^2} + \int_{k}^{\infty} \frac{1}{x^2} dx$. [6 marks]
- (d) Hence show that $\sum_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k+1} < L < \sum_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k}$. [2 marks]

You are given that $L = \frac{\pi^2}{6}$.

(e) By taking k = 4, use the upper bound and lower bound for L to find an upper bound and lower bound for π . Give your bounds to three significant figures. [3 marks]

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4. [Maximum mark: 18]

- (a) Use the limit comparison test to prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. [5 marks]
- (b) Express $\frac{1}{n(n+1)}$ in partial fractions and hence find the value of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. [4 marks]
- (c) Using the Maclaurin series for $\ln(1+x)$, show that the Maclaurin series for $(1+x)\ln(1+x)$ is $x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n+1}}{n(n+1)}$. [3 marks]
- (d) Hence find $\lim_{x \to -1} (1+x) \ln(1+x)$. [2 marks]
- (e) Write down $\lim_{x\to 0} x \ln(x)$. [1 mark]
- (f) Hence find $\lim_{x\to 0} x^x$. [3 marks]